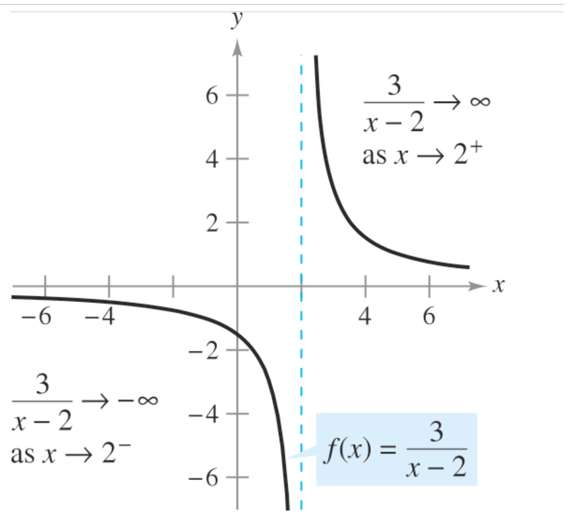
Math 150, *Lecture Notes- Bonds* **Name____________**

Section 1.5 *Infinite Limits*

Consider the function $f(x)=\dfrac{3}{x-2}$. In this case, we say that $f(x)$ ***decreases without bound as*** x ***approaches 2 from the left, and*** $f(x)$ ***increases without bound as*** x ***approaches 2 from the right.***



$f(x)$ increases and decreases without bound as x approaches 2.

x approaches 2 from the left. → ← x approaches 2 from the right.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	-6	-30	-300	-3000	?	3000	300	30	6

$f(x)$ decreases without bound. → ← $f(x)$ increases without bound.

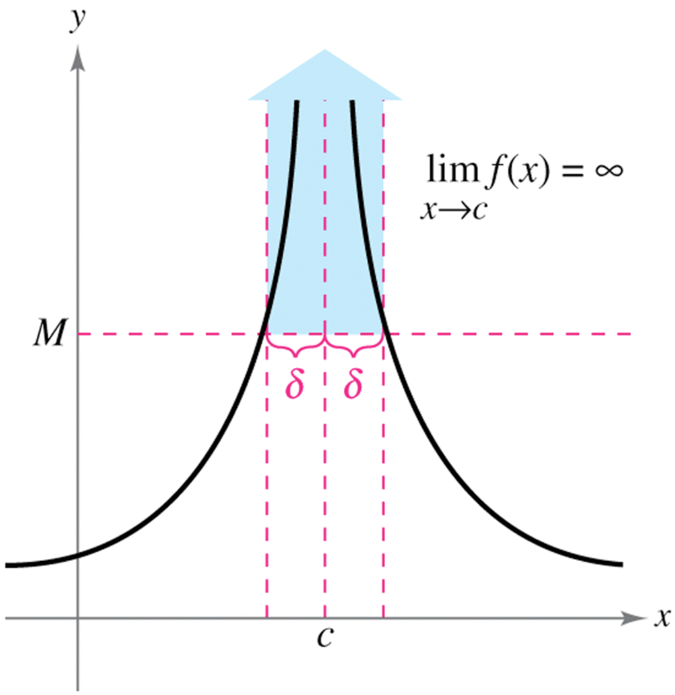
$$\lim_{x\to 2^-}\frac{3}{x-2}=-\infty$$

$f(x)$ decreases without bound as x approaches 2 from the left.

and

$$\lim_{x\to 2^+}\frac{3}{x-2}=\infty$$

$f(x)$ increases without bound as x approaches 2 from the right.



Definition of Infinite Limits

Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x\to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 1.40). Similarly, the statement

$$\lim_{x\to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$. To define the **infinite limit from the left,** replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right,** replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.

The symbols ∞ and $-\infty$ do not represent real numbers. They are convenient symbols used to describe unbounded conditions more concisely.

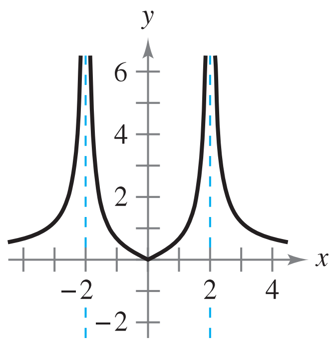
A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an **infinite limit**.

Be sure that you see the equal sign in the statement $\lim f(x) = \infty$ does not mean that the limit exists! On the contrary, it tells us ***how*** the limit **fails to exist** by denoting the unbounded behavior of $f(x)$ as x approaches c.

Determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -2 from the left and from the right.

Ex.1

$$f(x) = 2\left|\frac{x}{x^2 - 4}\right|$$



$\lim_{x \to -2} f(x) = \lim_{x \to -2} 2\left|\frac{x}{x^2-4}\right|$

$= \infty$

The function shows unbounded behavior upward on both sides of -2.

$\lim_{x \to -2^-} 2\left|\frac{x}{x^2-4}\right| = \infty$

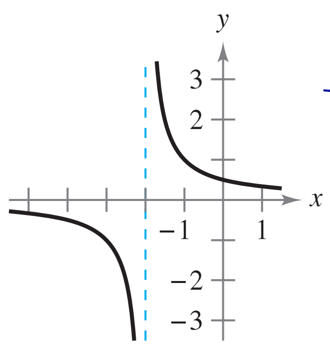
and

$\lim_{x \to -2^+} 2\left|\frac{x}{x^2-4}\right| = \infty$

These limits Do Not Exist.

Ex.2

$$f(x) = \frac{1}{x + 2}$$



$\lim_{x \to -2} \frac{1}{x+2}$ does not exist,

since $\lim_{x \to -2^-} \frac{1}{x+2} \neq \lim_{x \to -2^+} \frac{1}{x+2}$.

"Disagree"

$\lim_{x \to -2^-} \frac{1}{x+2} = -\infty$

&

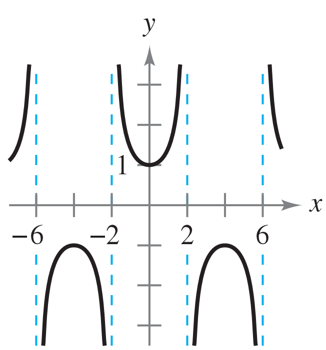
$\lim_{x \to -2^+} \frac{1}{x+2} = +\infty$

These limits do Not Exist.

We are seeing unbounded function behavior.

Ex.3

$$f(x) = \sec\frac{\pi x}{4}$$



$\lim_{x \to -2^-} \sec\left(\frac{\pi x}{4}\right) = -\infty$

$\lim_{x \to -2^+} \sec\left(\frac{\pi x}{4}\right) = +\infty$

$\lim_{x \to -2} \sec\left(\frac{\pi x}{4}\right)$ does not exist

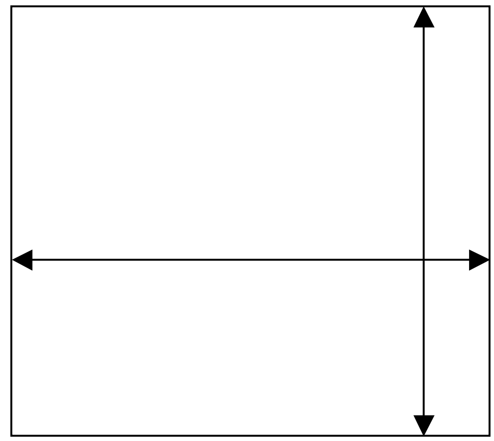
because $\lim_{x \to -2^-} \sec\left(\frac{\pi x}{4}\right) \neq \lim_{x \to -2^+} \sec\left(\frac{\pi x}{4}\right)$.

By completing the table, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -3 from the left and from the right. Graph the function to confirm your result.

Ex.4 $f(x)=\dfrac{x^2}{x^2-9}$

x	-3.5	-3.1	-3.01	-3.001
$f(x)$				

x	-2.999	-2.99	-2.9	-2.5
$f(x)$				



$\lim\limits_{x\to -3} \dfrac{x^2}{x^2-9}$ = "Does Not Exist"

"Disagree"

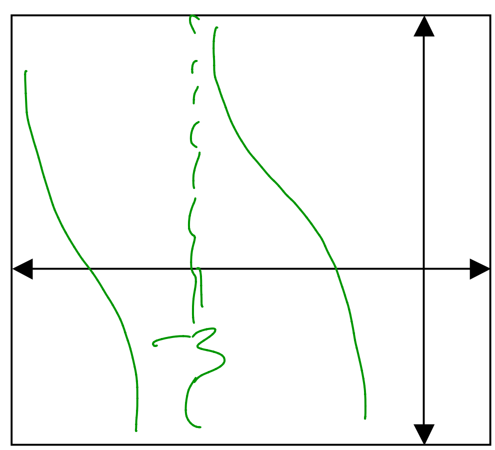
because

$$\lim_{x\to -3^-} \frac{x^2}{x^2-9} = +\infty \neq -\infty = \lim_{x\to -3^+} \frac{x^2}{x^2-9}$$

Ex.5 $f(x)=\cot\left(\dfrac{\pi x}{3}\right)$

x	-3.5	-3.1	-3.01	-3.001
$f(x)$				

x	-2.999	-2.99	-2.9	-2.5
$f(x)$				



$$\lim_{x\to -3^-} \cot\left(\frac{\pi x}{3}\right) = -\infty$$

$x=-3$

V.A.

$$\lim_{x\to -3^+} \cot\left(\frac{\pi x}{3}\right) = +\infty$$

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f.

THEOREM 1.14 Vertical Asymptotes

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

Find the vertical asymptotes of the graph of the function. Graph the function to confirm your result.

Ex.6 $f(x) = \dfrac{x^2-9}{x^3+3x^2-x-3}$

$f(x) = \dfrac{(x-3)(x+3)}{x^2(x+3)-1(x+3)}$

$f(x) = \dfrac{(x-3)(x+3)}{(x^2-1)(x+3)}$

$f(x) = \dfrac{(x+3)(x-3)}{(x+3)(x+1)(x-1)}$

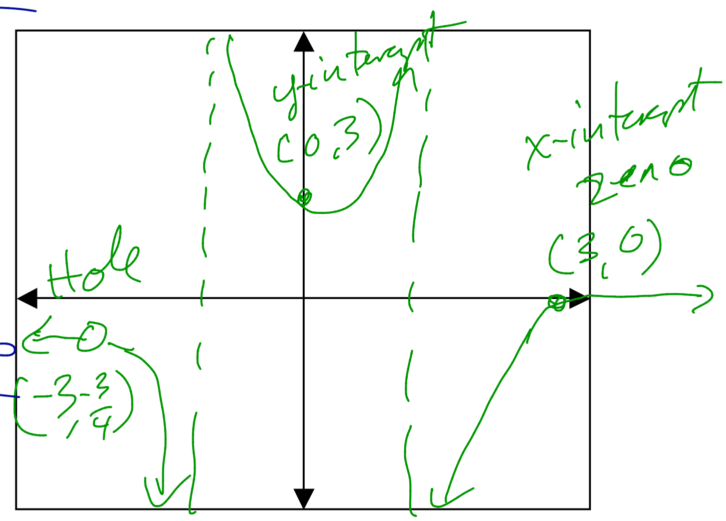
$x^3+3x^2-x-3=0$
$(x^2-1)(x+3)=0$
$x+3=0$
$x=-3$

$x-3 = (-3)-3$
$= -6 \neq 0$

hole at $x=-3$

$f(-3) = \dfrac{(-3)-3}{(-3+1)(-3-1)}$

$f(-3) = \dfrac{-6}{(-2)(-4)} = -\dfrac{3}{4}$



$x=-1$ V.A.

$x=1$ V.A.

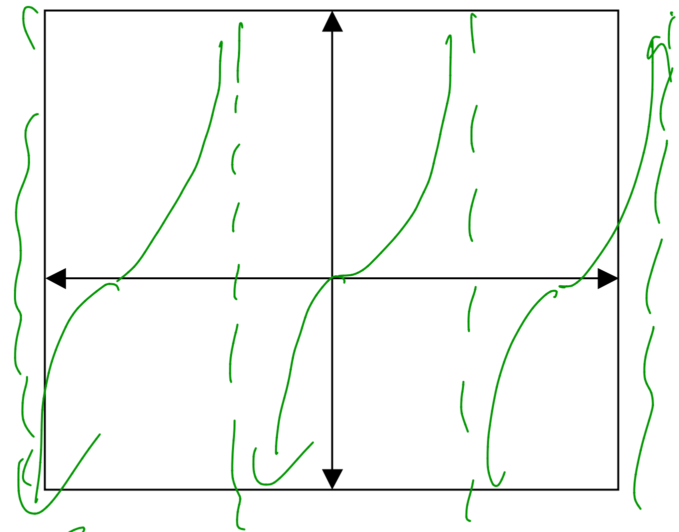
$(x^3+3x^2)-x-3 = x^2(x+3)-1(x+3)$ common factor
$= (x+3)(x^2-1)$

$A(B+C) = B\cdot A + C\cdot A$

$f(x) = \frac{\sin(\pi x)}{\cos(\pi x)}$ $\cos(\pi x) = 0$ $\sin(\pi(\frac{1}{2})) = \sin(\frac{\pi}{2}) = 1 \neq 0$

$\pi x = \frac{\pi}{2}$

Ex.7 $f(x) = \tan(\pi x)$ $x = \frac{1}{2}$



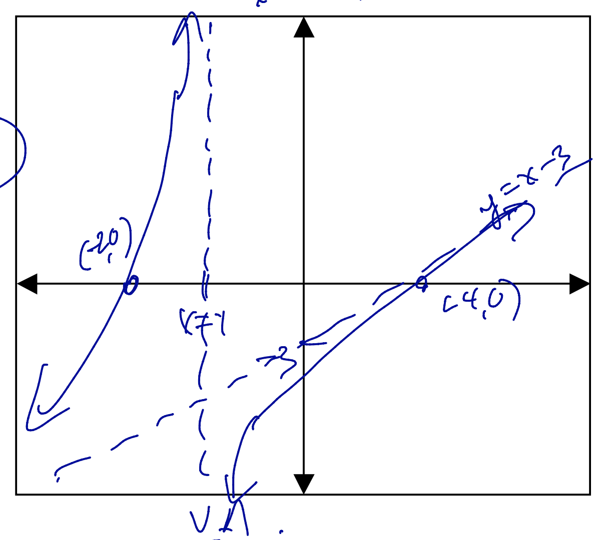
$\lim_{x \to \frac{1}{2}} \tan(\pi x) = \text{D.N.E.}$

$\lim_{x \to \frac{1}{2}^-} \tan(\pi x) = +\infty$

$\lim_{x \to \frac{1}{2}^+} \tan(\pi x) = -\infty$

Determine whether the function has a vertical asymptote, or a removable discontinuity at $x = -1$. Graph the function to confirm your result.

Ex.8 $f(x) = \frac{x^2 - 2x - 8}{x+1}$



$f(x) = \frac{(x+2)(x-4)}{(x+1)}$

$x+1=0$

$x = -1$

$(-1)^2 - 2(-1) - 8$
$= 1 + 2 - 8$
$= -5 \neq 0$

V.A. at $x=-1$

Non-Removable Discontinuity at $x=-1$

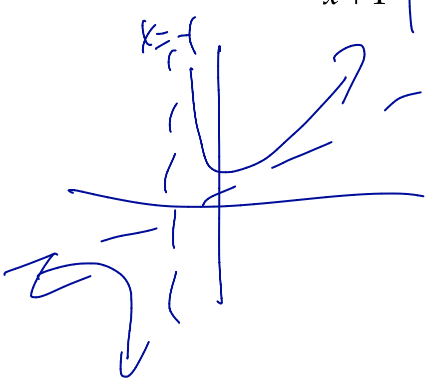
Slant Asymptote at $y = x-3$

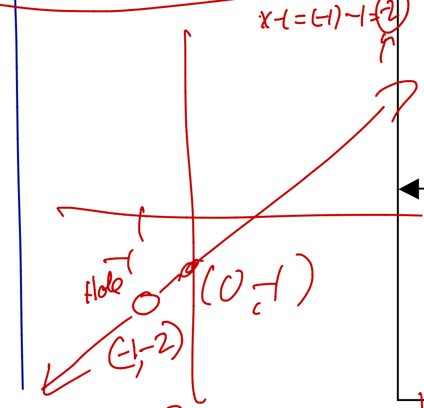
$x+1 \overline{) x^2 - 2x - 8}$ quotient $x - 3$
$-(x^2 + x)$
$-3x - 8$
$-3x - 3$

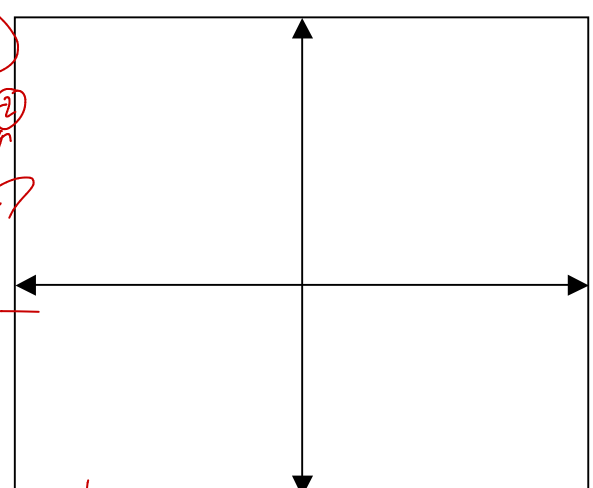
Ex.9 $f(x) = \frac{x^2+1}{x+1}$

$g(x) = \frac{x^2 - 1}{x+1} = \frac{(x+1)(x-1)}{x+1} = x - 1$

$x - 1 = (-1) - 1 = -2$







NonRemovable

Removable

THEOREM 1.15 Properties of Infinite Limits

Let c and L be real numbers and let f and g be functions such that

$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = L.$$

1. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = \infty$
2. Product: $\lim_{x\to c} [f(x)g(x)] = \infty, \quad L > 0$
   $\lim_{x\to c} [f(x)g(x)] = -\infty, \quad L < 0$
3. Quotient: $\lim_{x\to c} \frac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

Find the one-sided limit. If it does not exist, explain why.

Ex.10 $\lim_{x\to -\frac{1}{2}^+} \frac{6x^2+x-1}{4x^2-4x-3}$

Hole at $x = -\frac{1}{2}$

$2x+1=0$

$2x=-1$

$\boxed{x=-\frac{1}{2}}$

$= \lim_{x\to -\frac{1}{2}^+} \frac{(2x+1)(3x-1)}{(2x+1)(2x-3)}$

$= \lim_{x\to -\frac{1}{2}^+} \frac{(3x-1)}{(2x-3)}$

$= \frac{\lim_{x\to -\frac{1}{2}^+}(3x-1)}{\lim_{x\to -\frac{1}{2}^+}(2x-3)}$

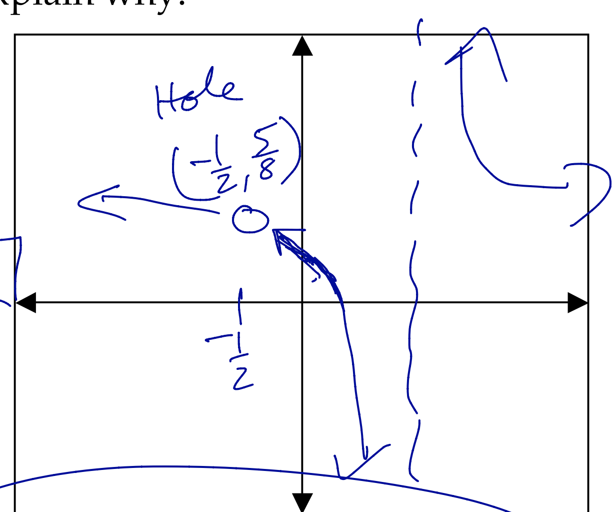
$= \frac{3(-\frac{1}{2})-1}{2(-\frac{1}{2})-3}$

$= \frac{-\frac{3}{2}-\frac{2}{2}}{-1-3}$

$= \frac{-\frac{5}{2}}{-4}$

$= \frac{5}{2}\cdot\frac{1}{4}$

$= \frac{5}{8}$



$2x-3=0$

$2x=3$

$x=\frac{3}{2}$

$x=\frac{3}{2}$

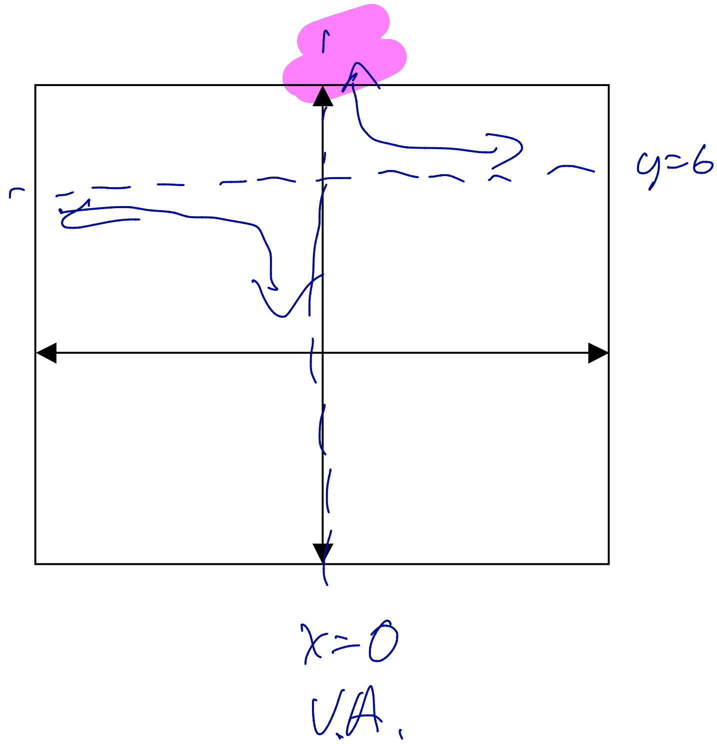
VA

Ex.11 $\lim_{x\to 0^+}\left(6+\frac{1}{x^3}\right)$

$= \lim_{x\to 0^+} 6 + \lim_{x\to 0^+} \frac{1}{x^3}$

$= 6 + \infty$

$= \infty$ D.N.E.



Ex.12 $\lim_{x\to 3^+}\left(\frac{x}{3}+\cot\left(\frac{\pi x}{2}\right)\right)$

$= \lim_{x\to 3^+} \frac{x}{3} + \lim_{x\to 3^+} \cot\left(\frac{\pi x}{2}\right)$

$= \frac{3}{3} + \lim_{x\to 3^+} \frac{\cos\left(\frac{\pi x}{2}\right)}{\sin\left(\frac{\pi x}{2}\right)}$

$= 1 + \frac{\lim_{x\to 3^+} \cos\left(\frac{\pi x}{2}\right)}{\lim_{x\to 3^+} \sin\left(\frac{\pi x}{2}\right)}$

$\cos\left(\frac{\pi x}{2}\right)=0$

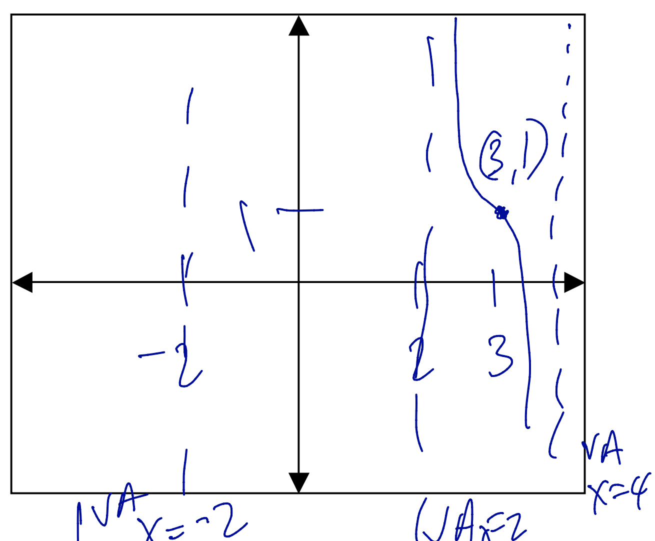
$\frac{\pi x}{2} = \frac{3\pi}{2}$

$x = 3$

$= 1 + \frac{0}{\sin\left(\frac{3\pi}{2}\right)}$

$= 1 + 0$

$= 1$



$\sin\left(\frac{\pi x}{2}\right) = 0$

$\frac{\pi x}{2} = 0$ OR $\frac{\pi}{2}x = \pi$

$x = 0$ $x = 2$

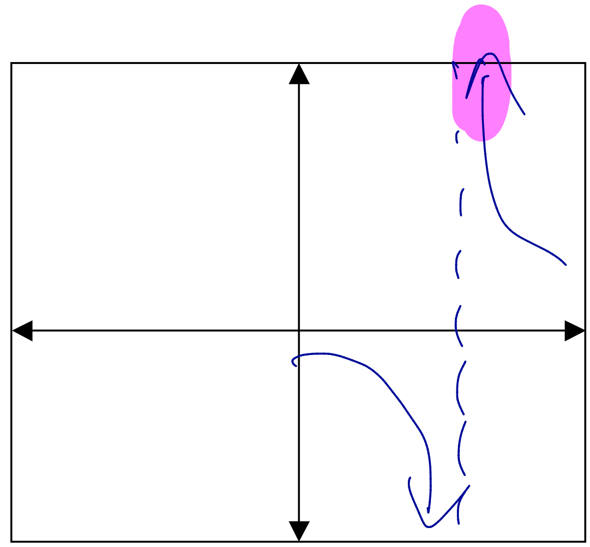
Ex.13 $\lim_{x\to\frac{\pi}{2}^+} \frac{-2}{\cos(x)} = +\infty$

$\cos\left(\frac{\pi}{2}\right) = 0$

$= \frac{\lim_{x\to\frac{\pi}{2}^+}(-2)}{\lim_{x\to\frac{\pi}{2}^+}\cos(x)}$ WRONG

$= \frac{-2}{0}$

$= +\infty$



$x = \frac{\pi}{2}$

$\cos\left(\frac{\pi}{2}\right)=0$ & $\cos\left(-\frac{\pi}{2}\right)=0$

VA $x=-\frac{\pi}{2}$ VA $x=\pi/2$

$\cot(x) = \frac{\cos(x)}{\sin(x)}$

Ex.14 $\lim_{x\to 0^-} \frac{x+2}{\cot(x)} = 0$

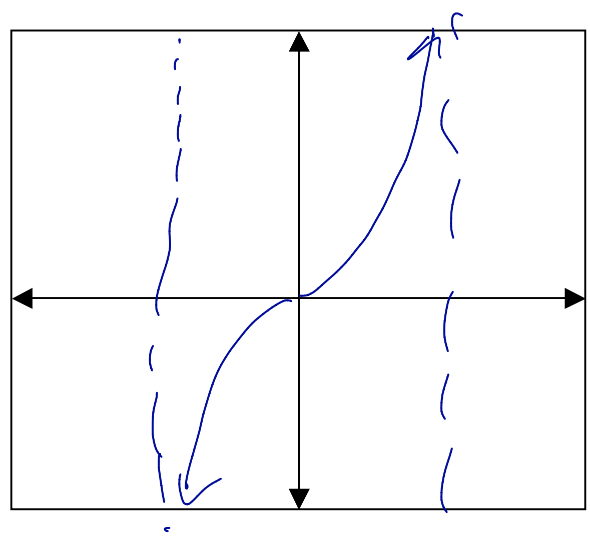
$= \lim_{x\to 0^-}(x+2) \cdot \lim_{x\to 0^-} \tan(x)$

$= (0+2) \cdot \lim_{x\to 0} \frac{\sin(x)}{\cos(x)}$

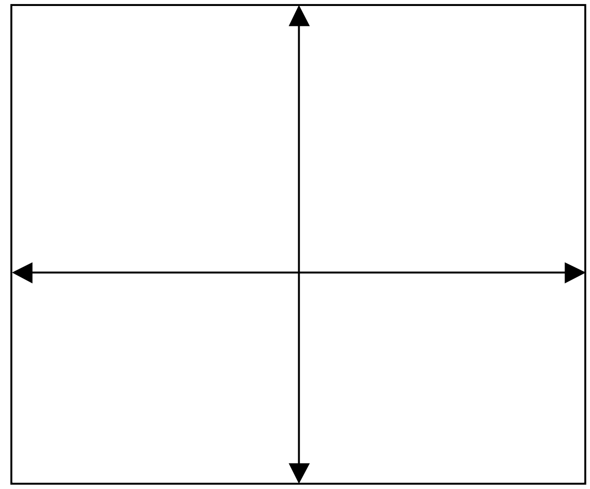
$= 2 \cdot \frac{\lim_{x\to 0} \sin(x)}{\lim_{x\to 0} \cos(x)}$

$= \frac{2\cdot 0}{1}$

$= 0$



Ex.15 $\lim_{x \to \frac{1}{2}^+} x^2 \tan(\pi x)$



Ex.15 $\lim\limits_{x \to \frac{1}{2}^+} x^2 \tan(\pi x)$